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Application of the finite element method with sectional linearization to flow problems

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SUMMARY
The application of the finite element method and Galerkin's principle to an open channel problem and a groundwater problem (determination of the boundary between fresh and salt water under an island in sea) is described. The pertaining differential equations are of nonlinear type. In order to avoid extensive computations the principle of sectional linearization of nonlinear expressions is introduced, yielding satisfactory results.

1. Introduction

The finite element method (f.e.m.) proves to be a powerful technique for solving boundary value problems, as has been shown by many publications. Most of the applications lie in the field of solid mechanics. However, recently the f.e.m. finds also acceptance in flow problems, see for instance [1]. Here the application of the f.e.m. in combination with Galerkin's principle to two such problems is treated.

The first one concerns nonstationary, one-dimensional, open channel flow, governed by a pair of nonlinear, first order differential equations, with the discharge and waterdepth as unknowns. Additional boundary conditions are given. By means of the f.e.m. the dependence of the variables with respect to the length coordinate is treated; a finite difference scheme is used in order to solve the resulting set of ordinary differential equations.

The second part of the paper deals with a groundwater problem: the determination of the stationary boundary between fresh and salt water under a circular island in sea. This problem has been treated earlier by the present author ([2]). Ordinary, nonlinear, second order differential equations were derived for the mentioned boundary, which were solved numerically, using Runge–Kutta's method.

It appeared that the latter technique was not very suitable for this case due to the occurrence of instabilities. Only after many efforts and the application of some tricks a satisfactory solution could be achieved. For that reason the differential equations are solved once again, but now by means of the f.e.m. Hence a somewhat unusual, but—as appears—successful employ of the f.e.m. is made.

Since the differential equations in both problems are of nonlinear type, the application of the f.e.m. leads to voluminous computations. In order to avoid extensive computational work, the principle of sectional linearization is introduced. The nonlinear expressions in the dependent variables, which are left after the application of the f.e.m. and Galerkin's principle, are linearized sectionally by means of a Taylor expansion. Then simplified equations are obtained. The corresponding loss of accuracy is mostly slight and vanishes if the number of elements increases. The principle is described in detail in the following sections.

2. Open channel flow

2.1. *Formulation of the problem*

A straight, horizontal channel with a uniform, trapezoidal cross section is considered (figure 1). The channel connects a marsh with a river and serves as a drainage-canal for the marsh. It is assumed that the marsh has a constant water-level and the river a periodical level. This periodicity is due to the influence of the tides; the period is approximately 12 hours. A sluice separates the canal from the river. Only when the water-level in the river is below that in the channel near the sluice, the sluice opens. The discharge is related to the waterdepth on both sides of the sluice: this relation is available in tabulated form. By virtue of the boundary conditions the channel flow will be a periodic function of time.

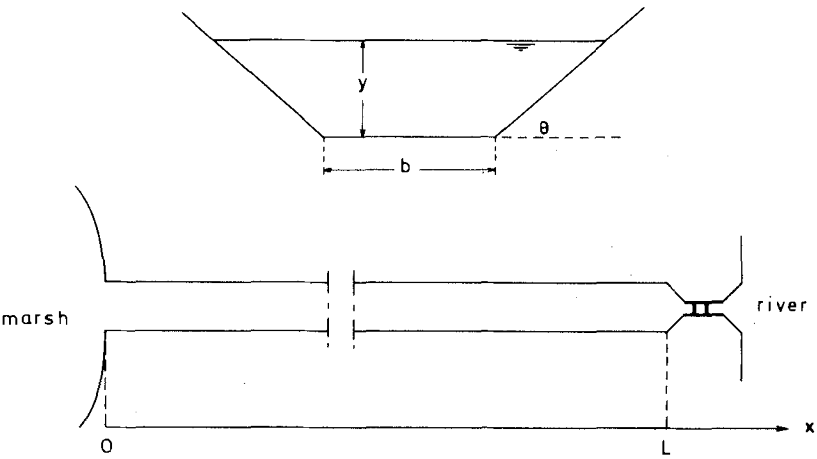

Figure 1. An outline and a cross section of the channel.

The complete differential equations governing the problem are given in the handbooks [3]–[6]. They may be written in the form

$$\frac{\partial q}{\partial x} + \frac{\partial A}{\partial t} = 0\,, \tag{2.1a}$$

$$\frac{q|q|O^{\frac{4}{3}}}{A^{10/3}k_m^2} + \left(1 - \frac{q^2}{gA^3}\frac{dA}{dy}\right)\frac{\partial y}{\partial x} + \frac{2q}{gA^2}\frac{\partial q}{\partial x} + \frac{1}{gA}\frac{\partial q}{\partial t} = 0\,. \tag{2.1b}$$

Here x represents the coordinate along the channel, t the time, q the discharge, y the waterdepth, A the wetted area, O the wetted perimeter, k_m the Manning coefficient (a friction coefficient depending on the overgrowth on the bottom and banks) and g the acceleration of gravity. A and O are functions of y only (figure 1), with

$$A = by + cy^2\,, \tag{2.2a}$$

$$O = b + dy\,, \tag{2.2b}$$

$$c = \operatorname{cotg}\theta\,, \tag{2.3a}$$

$$d = \frac{2}{\sin\theta}\,. \tag{2.3b}$$

The boundary condition at $x=0$ reads

$$y(0,t) = h\,, \tag{2.4}$$

where h represents the water-level of the marsh with respect to the channel bottom. At the other end of the channel, $x=L$, we have a relationship between the discharge $q_c = q(L,t)$, the waterdepth in the channel $h_c = h(L,t)$ and the water-level $h_r(t)$ in the river. This level is a periodic function of time,

$$h_r = h - a\sin\frac{2\pi t}{T}\,, \tag{2.5}$$

where the period T equals 12 hours. h_c is given as a function of q_c and h_r in tabulated form for $h_r < h_c$. For $h_r \geqq h_c$ we have as boundary condition $q_c = 0$. A quadratic interpolation formula of the form

$$h_c = a_1 + a_2 q_c + a_3 h_r + a_4 q_c h_r \tag{2.6}$$

is used between successive values of q_c and h_r in such a way that a continuous representation for h_c originates.

The equations (2.1) are written in non-dimensional form by introducing the quantities

$$\bar{x} = x/L, \ \bar{y} = y/h, \bar{q} = q/p, \ \bar{t} = t/T . \tag{2.7}$$

Here p is a unit of discharge, defined by

$$p = (h^3 b^2 g)^{\frac{1}{2}} . \tag{2.8}$$

With (2.7) the equations (2.1) transform into

$$\frac{\partial q}{\partial x} + \alpha \frac{\partial}{\partial t}(y + \gamma y^2) = 0 , \tag{2.9a}$$

$$\frac{\beta(1+\delta y)^{\frac{4}{3}} q|q|}{y^{\frac{1}{3}}(1+\gamma y)^{\frac{1}{3}}} + \{y^3(1+\gamma y)^3 - q^2(1+2\gamma y)\} \frac{\partial y}{\partial x}$$

$$+ 2qy(1+\gamma y) \frac{\partial q}{\partial x} + \alpha y^2 (1+\gamma y)^2 \frac{\partial q}{\partial t} = 0 . \tag{2.9b}$$

In (2.9), the bars are dropped. α, β, γ and δ are constants,

$$\alpha = \frac{bhL}{pT}, \ \beta = \frac{Lg}{h^{\frac{4}{3}} k_m^2}, \ \gamma = \frac{h}{b} \operatorname{cotg} \theta, \ \delta = \frac{2h}{b \sin \theta} . \tag{2.10}$$

The boundary conditions read

$$y(0, t) = 1 , \tag{2.11a}$$

$$h_c(t) = G\{h_r(t), q_c(t)\} , \quad h_r < h_c \tag{2.11b}$$

Condition (2.11b) represents the discharge relation for the sluice.

2.2. *The finite element method and the principle of sectional linearization*

According to the f.e.m. the range $0 \leqq x \leqq 1$ is divided into N subranges $[x_{n-1}, x_n]$, $n = 1 \ldots N$, with $x_0 = 0$ and $x_N = 1$. The length of the subrange $[x_{n-1}, x_n]$ is denoted by g_n. Further a set of sectionally linear shapefunctions $f_m(x)$, $m = 0 \ldots N$, is defined (figure 2), with $f_m(x_n) = \delta_{mn}$. Here δ_{mn} represents the Kronecker δ.

Applying Galerkin's principle with respect to $0 \leqq x \leqq 1$, a set of approximate equations is derived,

$$\int_0^1 \left\{ \frac{\partial q}{\partial x} + \alpha \frac{\partial}{\partial t}(y + \gamma y^2) \right\} f_m(x) dx = 0 , \quad m = 0 \ldots N , \tag{2.12a}$$

$$\int_0^1 \left[\frac{\beta(1+\delta y)^{\frac{4}{3}} q|q|}{y^{\frac{1}{3}}(1+\gamma y)^{\frac{1}{3}}} + \{y^3(1+\gamma y)^3 - q^2(1+2\gamma y)\} \frac{\partial y}{\partial x} \right.$$

$$\left. + 2qy(1+\gamma y) \frac{\partial q}{\partial x} + \alpha y^2 (1+\gamma y)^2 \frac{\partial q}{\partial t} \right] f_m(x) dx = 0 , \quad m = 0 \ldots N \tag{2.12b}$$

with $q(x, t)$ and $y(x, t)$ approximated by

$$q = \sum_{k=0}^{N} q_k(t) f_k(x) , \tag{2.13a}$$

$$y = \sum_{k=0}^{N} y_k(t) f_k(x) . \tag{2.13b}$$

By virtue of (2.11), two equations are dropped from the system (2.12). We may drop either the equation of the form (2.12a) for $m=0$ and (2.12b) for $m=N$, or (2.12a) for $m=N$ and (2.12b) for $m=0$. Here the first possibility is chosen. However the choice of the equations that are dropped does not effect the results significantly, as is confirmed by the computations.

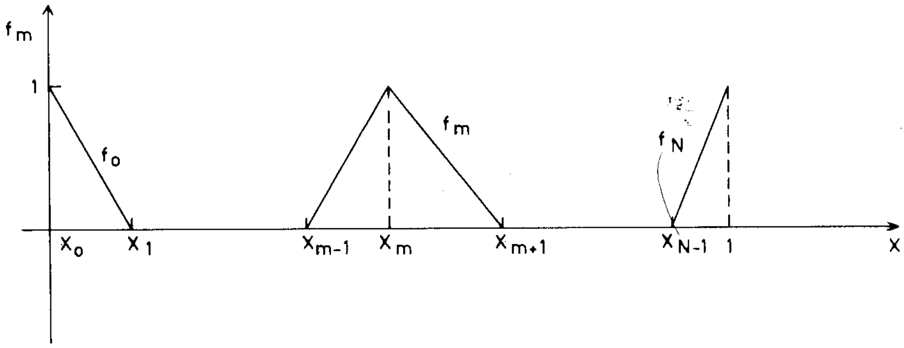

Figure 2. The elements and shape-functions.

Since the variables q and y in (2.12a) have small, positive, integer powers, the integrals in the left-hand side of these equations may be evaluated without any difficulty. We obtain

$$\tfrac{1}{2}(q_1 - q_0) + \frac{\alpha}{6}\frac{d}{dt} g_1 \left\{ y_1 + 2y_0 + \frac{\gamma}{2}(y_1^2 + 2y_1 y_0 + 3y_0^2) \right\} = 0 , \tag{2.14a}$$

$$\tfrac{1}{2}(q_{m+1} - q_{m-1}) + \frac{\alpha}{6}\frac{d}{dt}\Big[g_m(y_{m-1} + 2y_m) + g_{m+1}(y_{m+1} + 2y_m) +$$

$$\frac{\gamma}{2}\{ g_m(y_{m-1}^2 + 2y_m y_{m-1} + 3y_m^2) + g_{m+1}(y_{m+1}^2 + 2y_m y_{m+1} + 3y_m^2) \} \Big] = 0 ,$$

$$m = 1 \ldots N-1 , \tag{2.14b}$$

where $y_0 = 1$.

In (2.12b) fractional powers and larger, integer powers occur. In order to avoid extensive computations, the terms in the expression between square brackets in (2.12b) are linearized sectionally. If $s(y, q)$ is such a term, we linearize $s(y, q)$ in the interval $[x_{m-1}, x_{m+1}]$ in the following manner,

$$s(y, q) \approx s(y_m, q_m) + \frac{\partial s}{\partial y}(y_m, q_m)(y - y_m) + \frac{\partial s}{\partial q}(y_m, q_m)(q - q_m) . \tag{2.15}$$

From the approximations (2.13), we find for $x_{m-1} \leqq x \leqq x_m$,

$$y - y_m = (y_m - y_{m-1})(f_m - 1) , \tag{2.16a}$$

$$q - q_m = (q_m - q_{m-1})(f_m - 1) . \tag{2.16b}$$

Replacing the subscript $m-1$ by $m+1$, analogous expressions for $x_m \leqq x \leqq x_{m+1}$ are obtained.

By means of (2.15) and (2.16) the integrations in (2.12b) may be performed easily. The resulting equations read,

$$\frac{\beta(1+\delta y_m)^{\frac{4}{3}} q_m|q_m|}{18 y_m^{\frac{4}{3}}(1+\gamma y_m)^{\frac{4}{3}}}\left[g_m\left\{3+\frac{6q_{m-1}}{q_m}+(y_m-y_{m-1})\left(-\frac{4\delta}{1+\delta y_m}+\frac{1}{y_m}+\frac{\gamma}{1+\gamma y_m}\right)\right\}\right.$$
$$\left.+g_{m+1}\left\{3+\frac{6q_{m+1}}{q_m}+(y_m-y_{m+1})\left(-\frac{4\delta}{1+\delta y_m}+\frac{1}{y_m}+\frac{\gamma}{1+\gamma y_m}\right)\right\}\right]$$
$$+(y_m-y_{m-1})\left[\tfrac{1}{2}y_m^2(1+\gamma y_m)^2\{y_{m-1}(1+2\gamma y_m)-\gamma y_m^2\}-\tfrac{1}{6}\{(1+2\gamma y_m)\right.$$
$$\left.(q_m^2+2q_m q_{m-1})-2\gamma q_m^2(y_m-y_{m-1})\}\right]$$
$$+(y_{m+1}-y_m)\left[\tfrac{1}{2}y_m^2(1+\gamma y_m)^2\{y_{m+1}(1+2\gamma y_m)-\gamma y_m^2\}\right.$$
$$\left.-\tfrac{1}{6}\{(1+2\gamma y_m)(q_m^2+2q_m q_{m+1})-2\gamma q_m^2(y_m-y_{m+1})\}\right]$$
$$+\tfrac{1}{3}(q_m-q_{m-1})\{q_m y_{m-1}(1+2\gamma y_m)+q_{m-1}y_m(1+\gamma y_m)+q_m y_m\}$$
$$+\tfrac{1}{3}(q_{m+1}-q_m)\{q_m y_{m+1}(1+2\gamma y_m)+q_{m+1}y_m(1+\gamma y_m)+q_m y_m\}$$
$$+\alpha g_m\left[\tfrac{1}{3}y_m(1+\gamma y_m)\{(1+2\gamma y_m)y_{m-1}-\gamma y_m^2\}\frac{dq_m}{dt}+\tfrac{1}{6}y_m^2(1+\gamma y_m)^2\frac{dq_{m-1}}{dt}\right]$$
$$+\alpha g_{m+1}\left[\tfrac{1}{3}y_m(1+\gamma y_m)\{(1+2\gamma y_m)y_{m+1}-\gamma y_m^2\}\frac{dq_m}{dt}\right.$$
$$\left.+\tfrac{1}{2}y_m^2(1+\gamma y_m)^2\frac{dq_{m+1}}{dt}\right]=0\,,\qquad m=1\ldots N-1\,, \tag{2.17a}$$

$$\frac{\beta g_N(1+\delta y_N)^{\frac{4}{3}} q_N|q_N|}{18 y_N^{\frac{4}{3}}(1+\gamma y_N)^{\frac{4}{3}}}\left\{3+\frac{6q_{N-1}}{q_N}+(y_N-y_{N-1})\left(-\frac{4\delta}{1+\delta y_N}+\frac{1}{y_N}+\frac{\gamma}{1+\gamma y_N}\right)\right\}$$
$$+(y_N-y_{N-1})\left[\tfrac{1}{2}y_N^2(1+\gamma y_N)^2\{y_{N-1}(1+2\gamma y_N)-\gamma y_N^2\}-\tfrac{1}{6}\{(1+2\gamma y_N)\right.$$
$$\left.(q_N^2+2q_N q_{N-1})-2\gamma q_N^2(y_N-y_{N-1})\}\right]$$
$$+\tfrac{1}{3}(q_N-q_{N-1})\{q_N y_{N-1}(1+2\gamma y_N)+q_{N-1}y_N(1+\gamma y_N)+q_N y_N\}$$
$$+\alpha g_N\left[\tfrac{1}{3}y_N(1+\gamma y_N)\{(1+2\gamma y_N)y_{N-1}\right.$$
$$\left.-\gamma y_N^2\}\frac{dq_N}{dt}+\tfrac{1}{6}y_N^2(1+\gamma y_N)^2\frac{dq_{N-1}}{dt}\right]=0\,. \tag{2.17b}$$

The differential equations (2.14) and (2.17) are solved simultaneously by means of an implicit, finite difference scheme, see [7]. Therefore the range $t\geqq 0$ is divided into a number of subranges $[t_i, t_{i-1}]$, $i=1, 2, \ldots$, which all have an equal length Δt. A set of $2N+1$ nonlinear equations results from (2.11), (2.14) and (2.17) for each time-interval $[t_i, t_{i-1}]$ with $q_n(t_i)$, $i=0 \ldots N$ and $y_n(t_i)$, $i=1 \ldots N$ as unknowns. These equations are solved by means of Newton–Raphson's technique. The solution for $t=t_{i-1}$ is the first approximation in the iteration scheme.

2.3. *Numerical results*

Numerical computations are performed for $h=1.4\,m$, $a=0.5\,m$ and $\theta=\text{arctg}\;0.5$. The remaining parameters, L, b and k_m, are variable. The computations are started with the constant initial situation $y_m=1$ and $q_m=0$, $m=0\ldots N$, and continued until a purely periodic behaviour is achieved.

It appears that for the cases considered a sufficient accuracy (three or more significant digits in y_m and q_m) exists for $\Delta t=T/80$ and $N=20$. The intervals $[x_{n-1}, x_n]$ have an equal length. Some results concerning the interesting quantity $q(x, t)$ are presented in the following plots. In figure 3, the discharge for $x=0$, near the marsh, and the discharge through the sluice, denoted respectively by $q(0)$ and $q(L)$, are given during a period for 3 values of L. Here fixed values for

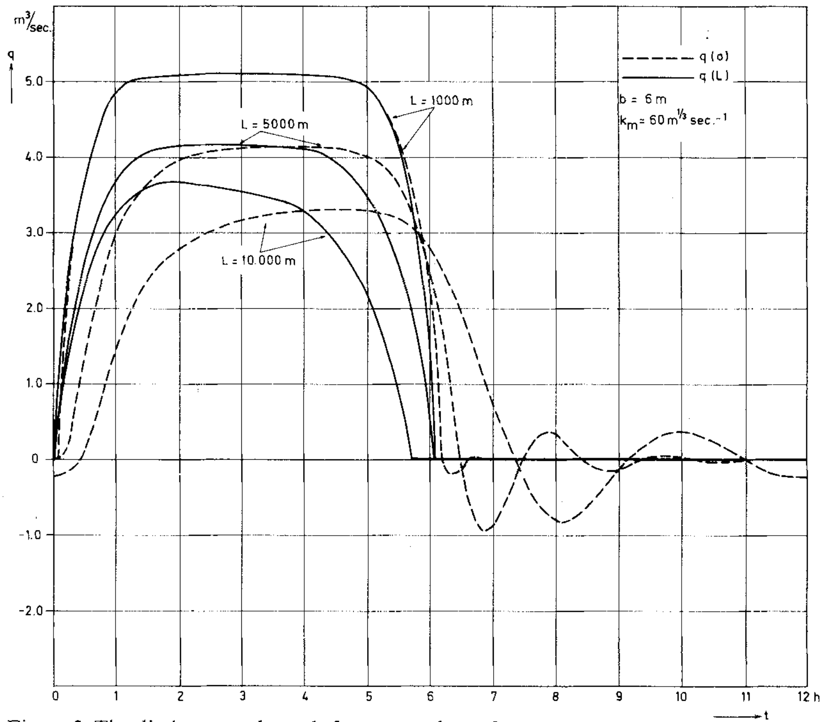

Figure 3. The discharge at the ends for some values of L.

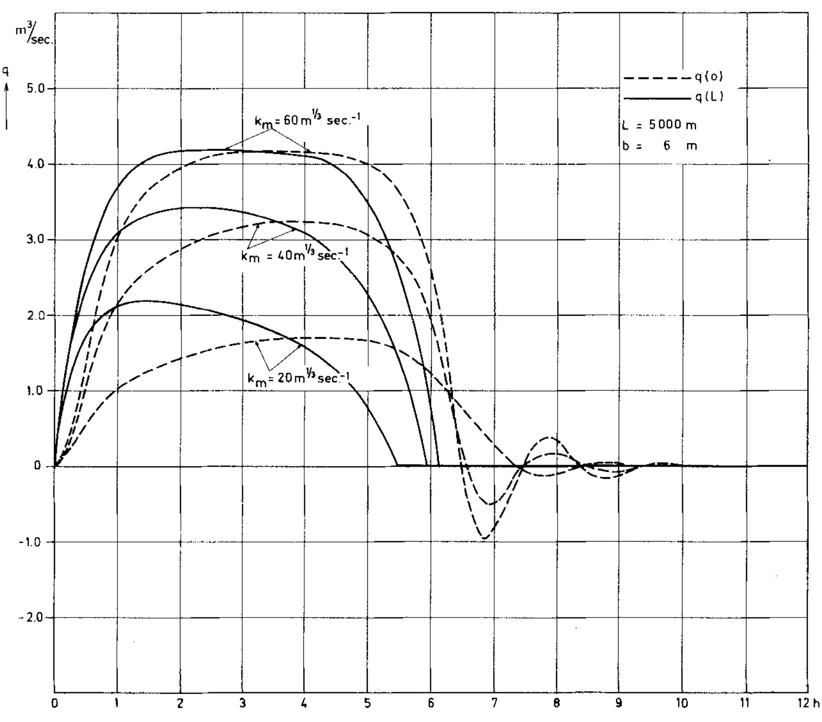

Figure 4. The discharge at the ends for some values of k_m.

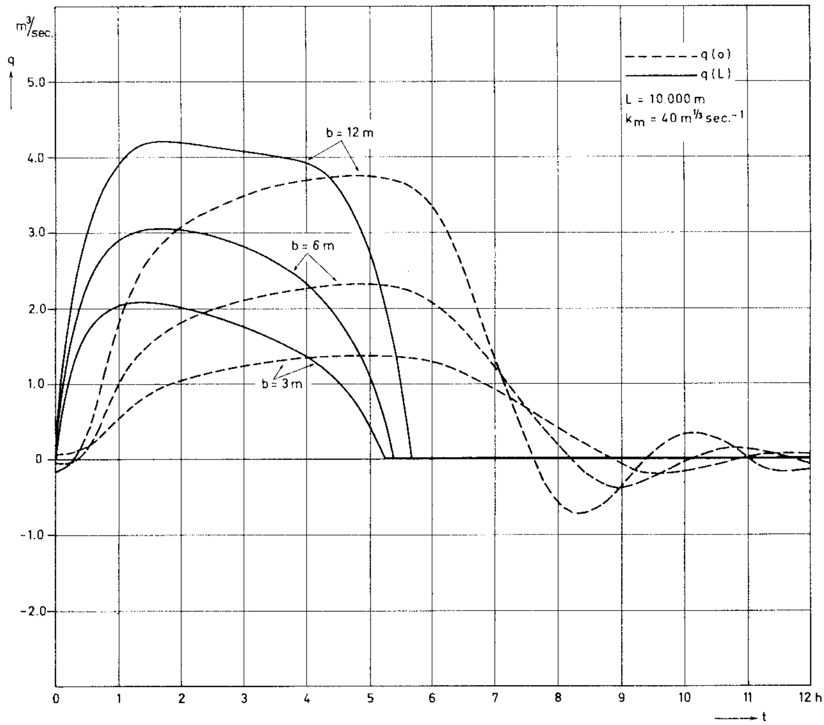

Figure 5. The discharge at the ends for some values of b.

b and k_m are chosen. Figure 4 presents these quantities for 3 values of k_m and fixed L and b. At last in figure 5 results are given for fixed L and k_m and 3 values of b.

In spite of the nonlinearity of the equations and the complexity of the boundary condition near the sluice, no problems occur with respect to the computations. For increasing N and decreasing Δt, the solutions for q_m and y_m converge to limit functions, showing the expected physical phenomena. For all values of t the wavedepth and discharge in the channel are smooth functions without disturbances due to numerical instability.

3. A groundwater problem

3.1. *Formulation of the problem*

We consider a circular island in sea. A cross section of the configuration is given in figure 6, which is copied from [2]. In this reference also a complete description is given.

For each of the regions $0 \leqq r \leqq b$, $b \leqq r \leqq 1$ and $1 \leqq r \leqq d$, a nonlinear ordinary differential equation was derived. In dimensionless form they are given by (5.2)–(5.4) in [2]. With the substitution $h = -y^{\frac{1}{2}}$, these equations transform into

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) - \frac{2r}{c}(z+y)^{\frac{1}{2}} = 0\,, \qquad 1 \leqq r \leqq d\,, \tag{3.1a}$$

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) - \frac{2r}{\gamma c}\{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} = 0\,, \quad b \leqq r \leqq 1\,, \tag{3.1b}$$

$$\frac{d}{dr}\left(r\frac{dy}{c}\right) - \frac{2r}{\gamma c}\{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (E - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} = 0\,, \qquad 0 \leqq r \leqq b\,,. \tag{3.1c}$$

where

$$D = z^2 + \gamma y(1) + \tfrac{1}{2}\rho\,, \tag{3.2a}$$

$$E = D + Q\rho \ln b\,, \tag{3.2b}$$

and b, c, z, γ and ρ are constants. The radius d is unknown.

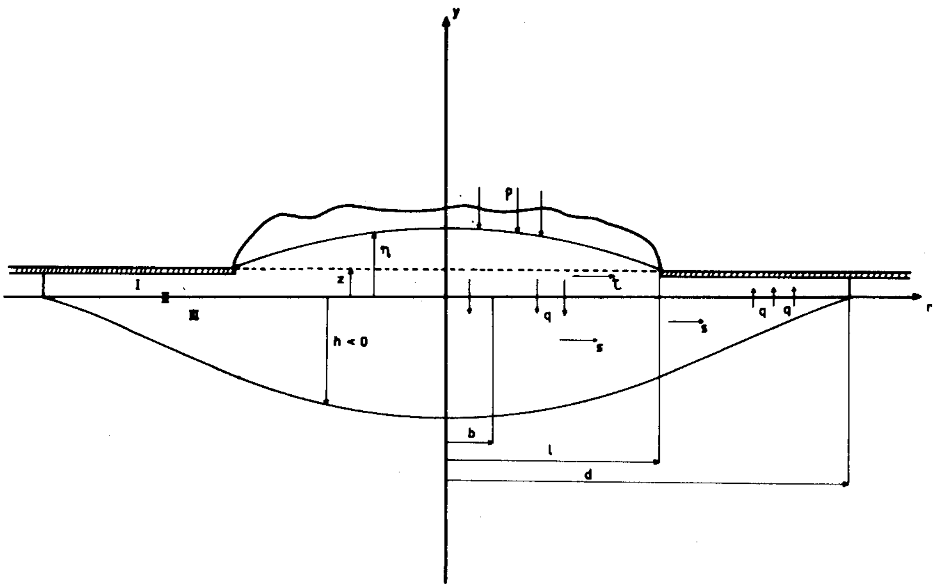

Figure 6. Cross section of the island.

The boundary conditions read

$$r = d\,,\ y = 0\,,\ \frac{dy}{dr} = 0\,, \tag{3.3a}$$

$$r = 1\,,\ y(1^+) = y(1^-)\,,\ \frac{dy}{dr}(1^+) = \frac{dy}{dr}(1^-)\,, \tag{3.3b}$$

$$r = b\,,\ y(b^+) = y(b^-)\,,\ \frac{dy}{dr}(b^+) = \frac{dy}{dr}(b^-) + \frac{Q\rho}{\gamma b}\,, \tag{3.3c}$$

$$r = 0\,,\ \frac{dy}{dr} = 0\,. \tag{3.3d}$$

3.2. *The finite element method and the principle of sectional linearization*

The range $[0, d]$ is divided into N subintervals $[r_{n-1}, r_n]$, $n = 1 \dots N$, in such a way that $r_0 = 0$, $r_K = b$, $r_L = 1$ and $r_N = d$. K, L and N are integers such that $K < L < N$. Again a set of shape-functions $f_m(r)$ is introduced, with $f_m(r_n) = \delta_{mn}$. These functions will be sectionally linear on the interval $[0, r_{N-1}]$. In order to accommodate condition (3.3a) a quadratic representation for f_{N-1} and f_N in $[r_{N-1}, d]$ is chosen.

By means of Galerkin's principle and partial integration, we derive from (3.1) and (3.3),

$$\int_{r_{m-1}}^{r_{m+1}} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \alpha r f_m \{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (E - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\}\right] dr = 0\,,$$

$$m = 0 \dots K-1\,, \tag{3.4a}$$

$$\int_{r_{K-1}}^{r_K} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \alpha r f_m \{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (E - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} \right] dr$$
$$+ \int_{r_K}^{r_{K+1}} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \alpha r f_m \{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} \right] dr$$
$$= -\frac{Q\rho}{\gamma}, \quad (3.4b)$$

$$\int_{r_{m-1}}^{r_{m+1}} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \alpha r f_m \{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} \right] dr = 0, \qquad m = K+1, \dots L-1, \quad (3.4c)$$

$$\int_{r_{L-1}}^{r_L} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \alpha r f_m \{\gamma \cdot y^{\frac{1}{2}} + (\gamma+1)z - (D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}}\} \right] dr$$
$$+ \int_{r_L}^{r_{L+1}} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \frac{2r}{c} f_m (z + y^{\frac{1}{2}}) \right] dr = 0, \quad (3.4d)$$

$$\int_{r_{m-1}}^{r_{m+1}} \left[r\frac{dy}{dr}\frac{df_m}{dr} + \frac{2r}{c} f_m (z + y^{\frac{1}{2}}) \right] dr = 0, \qquad m = L+1, \dots N. \quad (3.4e)$$

where $\alpha = 2/(\gamma c)$.

For $m=0$ the lower bound r_{m-1} in (3.4a) must be replaced by 0 and for $m=N$ the upper bound r_{m+1} in (3.4e) by d. The unknown $y(r)$ is approximated by

$$y = \sum_{k=0}^{N} y_k f_k(r). \quad (3.5)$$

The nonlinear terms in the left-hand sides of (3.4) are linearized sectionally as far as necessary in order to evaluate the integrals in (3.4) analytically. Hence we use for $[r_{m-1}, r_{m+1}]$ the approximations

$$y^{\frac{1}{2}} \approx (y_m)^{\frac{1}{2}} \left(1 + \frac{y - y_m}{2y_m} \right), \quad (3.6a)$$

$$(E - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}} \approx (E_m)^{\frac{1}{2}} \left\{ 1 - \frac{\tfrac{1}{2}\rho (r^2 - r_m^2) + \gamma (y - y_m)}{2(E_m)^{\frac{1}{2}}} \right\}, \quad (3.6b)$$

$$(D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma y)^{\frac{1}{2}} \approx$$
$$\approx (D_m)^{\frac{1}{2}} \left\{ 1 + \frac{\frac{Q\rho}{r_m}(r - r_m) - \tfrac{1}{2}\rho (r^2 - r_m^2) + \gamma (y - y_m)}{2(D_m)^{\frac{1}{2}}} \right\}, \quad (3.6c)$$

where

$$E_m = z^2 + \gamma (y_L - y_m) + \tfrac{1}{2}\rho (1 - r_m^2) + Q\rho \ln r_K, \quad (3.7a)$$

$$D_m = z^2 + \gamma (y_L - y_m) + \tfrac{1}{2}\rho (1 - r_m^2) + Q\rho \ln r_m. \quad (3.7b)$$

Substituting (3.6) into (3.4), the integrals may be calculated easily for $[0, r_{N-1}]$.

The equations for the interval $[r_{N-1}, r_N]$ are treated in a different manner. The shape-function f_{N-1} is defined by

$$f_{N-1} = \left(\frac{r - d}{g_N} \right)^2, \qquad r_{N-1} \leqq r \leqq r_N. \quad (3.8)$$

Since $y_N = 0$, we have the following approximation for y in the interval $[r_{N-1}, r_N]$,

$$y \approx y_{N-1} \left(\frac{r - d}{g_N} \right)^2. \quad (3.9)$$

The shape-function f_N is defined as

$$f_N = 1 - f_{N-1}\,, \qquad r_{N-1} \leqq r \leqq r_N\,. \tag{3.10}$$

Using (3.9) and (3.10), the integrations for $[r_{N-1}, r_N]$ may be performed without employing the principle of sectional linearization.

Evaluating the integrals as described above, we obtain a system of $N+1$ nonlinear algebraic equations. The unknowns are y_m, $m=0 \ldots N-1$ and the radius of the bubble, d. The interval lengths g_n are chosen as follows,

$$g_n = \begin{cases} \dfrac{b}{K}, & n = 1 \ldots K\,, \\ \dfrac{1-b}{L-K}, & n = K+1, \ldots, L\,, \\ \dfrac{d-g_N-1}{N-L-1}, & n = L+1, \ldots, N-1\,, \end{cases}$$

where g_N is a constant which is independent of d.

The resulting equations are solved by means of Newton–Raphson's technique.

3.3. *Numerical results*

The depth of the boundary between the fresh and salt water is computed for $Q=0$, 0.0375 and 0.1125. For the remaining parameters we take $c=12.5$, $\gamma=0.024$, $b=0.1$, $z=0.0075$ and $\rho=0.00017$. Also for these values Runge–Kutta's method was applied, see [2].

Then initial values for y_m and d for Newton–Raphson's scheme have to be chosen. In virtue of the experiences with this problem, a large sensitivity with respect to the choice of these values should not be unrealistic. That is, we expect that no convergence to the physically realistic solution will exist, if the initial values differ too much from it. However, computations show the contrary: the iteration scheme is rather insensitive for the initial values. For instance, starting with the analytical solution for $c=0$ as given in [2], which is modified somewhat for $r>0.8$, a convergent scheme is obtained for all mentioned values of Q. The solutions achieved agree perfectly with the ones given in [2] for $K=4$, $L=14$ and $N=24$. They are shown by figure 5.1 in this reference.

It is striking that the f.e.m. yields the same solutions as Runge–Kutta's method, without exhibiting the numerical problems associated with the latter technique. Hence the computations performed confirm the power of the f.e.m.

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